JOURNAL OF APPROXIMATION THEORY 66, 279–287 (1991)

L^p-Inverse Theorems for Beta Operators*

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Communicated by Vilmos Totik

Received October 30, 1989; revised June 13, 1990

In this paper we give the direct and inverse theorems for beta operators. Some other approximation properties of these operators are also given.

1. INTRODUCTION

Beta transform was introduced for studying conditions for the regularity of sequence-to-sequence transformations. Beta operators are given by

$$\beta_n(f, x) = \int_0^\infty b_n(x, u)\, f(1/u)\, du \qquad (x > 0), \tag{1.1}$$

where $f \in L^p[0, \infty)$ $(1 \leqslant p \leqslant \infty)$, and

$$b_n(x, u) = \frac{x^n}{B(n, n)} \frac{u^{n-1}}{(1+xu)^{2n}}, \tag{1.2}$$

$$B(n, n) = ((n-1)!)^2/(2n-1)!. \tag{1.3}$$

Some papers [19, 20] have considered the approximation properties of beta operators. In this paper we give the direct and inverse theorem for these operators in $L^p[0, \infty)$ $(1 \leqslant p \leqslant \infty)$.

For consideration of the connection between the rate of approximation and the smoothness of the functions we use the modulus of smoothness given by Z. Ditzian and V. Totik [9], which in our result is given by

$$\omega_\varphi^2(f, t)_p = \sup_{0 < h \leqslant t} \|\Delta_{h\varphi}^2 f\|_{L^p[0,\infty)}, \tag{1.4}$$

* Supported by the National Science Foundation of China.

0021-9045/91 $3.00

where $f \in L^p[0, \infty)$, $\varphi(x) = x$, and

$$\Delta_h^2 f(x) = f(x+h) - 2f(x) + f(x-h) \qquad \text{for} \quad x \in [h, \infty);$$
$$\Delta_h^2 f(x) = 0, \quad \text{otherwise.}$$

This modified modulus of smoothness has the known properties of moduli of smoothness which were found by V. Totik [15]. It is a valuable tool for dealing with the rate of approximation, inverse theorems, and imbedding theorems.

We use the technique of interpolation spaces and characterization of K-functional which has been used for inverse theorems by a number of mathematicians [4, 5, 8, 9, 14].

Now let us give some notations.

For $1 \leqslant p \leqslant \infty$, let

$$D = \{g\colon g \in L^p[0, \infty),\ g' \text{ absolutely continuous locally},\ \varphi^2 g'' \in L^p[0, \infty)\} \tag{1.5}$$

be the weighted Sobolev space, and define

$$S(g) = \varphi^2 g''; \qquad \|S(g)\|_p = \|\varphi^2 g''\|_{L^p[0,\infty)}. \tag{1.6}$$

For $f \in L^p[0, \infty)$, let

$$K_{\varphi,2}(f, t)_p = \inf_{g \in D} \{\|f - g\|_p + t\|S(g)\|_p\} \qquad (t > 0) \tag{1.7}$$

be the so-called K-functional.

Z. Ditzian and V. Totik proved (see [9, Chaps. 1–3]) the interpolation theorem, that is, the equivalence of this K-functional to the above modulus of smoothness.

Now we can give our direct and inverse theorem for beta operators as follows.

THEOREM 1. *For* $f \in L^p[0, \infty)$, $1 \leqslant p \leqslant \infty$, $0 < \alpha < 1$, *the following statements are equivalent*:

(1) $\|\beta_n(f) - f\|_p = O(n^{-\alpha})$; (1.8)

(2) $K_{\varphi,2}(f, t)_p = O(t^\alpha)$; (1.9)

(3) $\|\Delta_{h\varphi}^2 f\|_{L^p[0,\infty)} = O(h^{2\alpha})$; (1.10)

(5) $\omega_\varphi^2(f, t)_p = O(t^{2\alpha})$. (1.11)

For $p = \infty$ we restrict our discussion to continuous functions.

Remark 1. It is easy to prove by the methods in [10, 12, 13] that our theorem is still valid for $\alpha = 1$.

Remark 2. The L^∞ case of our theorem can be derived from a result of V. Totik [16] by a minor change.

The results in Section 2 show that beta operators have some interesting properties.

2. Lemmas

First we give some formulae and estimates. For $1 < p < \infty$, define $q := p/(p-1)$.

Lemma 2.1. *Beta operators have the following properties*:

$$\beta_n(\ln t, x) = \ln x; \tag{2.1}$$

$$\beta_n(t^m, x) = B(n+m, n-m)\, x^m/B(n, n), \tag{2.2}$$

for $m \in Z$, $m < n$, *where* $B(n+m, n-m) = (n+m-1)!(n-m-1)!/(2n-1)!$. *In particular, we have for* $n > 2$

$$\beta_n(t, x) = nx/(n-1); \tag{2.3}$$

$$\beta_n(t^2, x) = n(n+1)\, x^2/((n-1)(n-2)); \tag{2.4}$$

$$\beta_n(1/t, x) = n/((n-1)x). \tag{2.5}$$

Proof. For $m < n$, we have

$$\begin{aligned} \beta_n(t^m, x) &= \int_0^\infty \frac{1}{B(n, n)} \frac{t^{n-1-m}}{(1+t)^{2n}}\, dt\, x^m \\ &= B(n+m, n-m)\, x^m/B(n, n). \end{aligned}$$

In the same way we have

$$\begin{aligned} \beta_n(\ln t, x) - \ln x &= \beta_n(\ln(t/x), x) \\ &= -\int_0^\infty \frac{1}{B(n,n)} \frac{t^{n-1}}{(1+t)^{2n}} \ln t\, dt \\ &= 0. \end{aligned}$$

Lemma 2.2. *For* $1 \leqslant p \leqslant \infty$, $f \in D$, *we have*

$$\|\varphi f'\|_p \leqslant \|\varphi^2 f''\|_p. \tag{2.6}$$

Proof. First let us prove that $\lim_{x\to\infty} f'(x)=0$ for $f\in D$.
For $p=1$, $x, y>0$, we have

$$|f'(x)-f'(y)| = \left|\int_x^y u^2 f''(u)\, u^{-2}\, du\right| \leqslant \|\varphi^2 f''\|_1/\min\{x^2, y^2\}.$$

For $p=\infty$, we have

$$|f'(x)-f'(y)| \leqslant \|\varphi^2 f''\|_\infty/\min\{x, y\}.$$

For $1<p<\infty$, we have

$$|f'(x)-f'(y)| \leqslant \left(\left|\int_x^y |u^2 f''(u)|^p\, du\right|\right)^{1/p} \left(\left|\int_x^y u^{-2q}\, du\right|\right)^{1/q}$$
$$\leqslant \|\varphi^2 f''\|_p (2q-1)^{-1/q} (\min\{x, y\})^{(1-2q)/q}$$

Thus we know that $\lim_{x,y\to\infty} |f'(x)-f'(y)|=0$ and $\lim_{x\to\infty} f'(x)$ exists, but the condition $f\in L^p[0,\infty)$ implies $\lim_{x\to\infty} f'(x)=0$.

Now the proof of our lemma can be derived easily from the Hardy inequality for $1\leqslant p<\infty$, $g\geqslant 0$, and $\beta>0$ given by

$$\left(\int_0^\infty \left(\int_x^\infty g(y)\, dy\right)^p x^{\beta-1}\, dx\right)^{1/p} \leqslant p \left(\int_0^\infty \left(y g(y)\right)^p y^{\beta-1}\, dy\right)^{1/p} \Big/ \beta. \tag{2.7}$$

The case $p=\infty$ is simpler and our proof is complete.

LEMMA 2.3. *For $1\leqslant p\leqslant\infty$, $n\geqslant 2$, we have*

$$\|\beta_n\|_p \leqslant n/(n-1). \tag{2.8}$$

Proof. For $1<p<\infty$, $f\in L^p[0,\infty)$, we have

$$\|\beta_n(f)\|_p^p \leqslant \int_0^\infty \int_0^\infty b_n(x, t)\, |f(1/t)|^p\, dt\, dx$$
$$= \int_0^\infty |f(1/t)|^p\, dt \int_0^\infty b_n(x, t)\, dx$$
$$= n\|f\|_p^p/(n-1).$$

The proofs in the cases $p=1$ and ∞ are trivial.

The following property of beta operators is very interesting.

LEMMA 2.4. *Suppose $r \in N$, $f^{(r-1)}$ is absolutely continuous locally, and $\varphi^r f^{(r)} \in L^p[0, \infty)$. We have*

$$\varphi^r \beta_n^{(r)}(f) = \beta_n(\varphi^r f^{(r)}). \tag{2.9}$$

Proof. Note that

$$\beta_n(f, x) = \int_0^\infty \frac{v^{n-1}}{B(n, n)(1+v)^{2n}} f(x/v)\, dv. \tag{2.10}$$

We can differentiate below the integration sign on $[c, +\infty)$ for any $c > 0$ since the integration formula

$$x^{-r-1} \int_0^\infty \frac{v^{n-3}}{B(n, n)(1+v)^{2n}} \frac{x}{v^2} (x/v)^r f^{(r)}(x/v)\, dv$$

is uniformly convergent on $[c, +\infty)$, hence our proof is trivial.

With all the above preparations we can now prove our theorem.

3. PROOF OF THE INVERSE THEOREM

From the following lemmas we know that (1) and (2) are equivalent by a result of A. Grundmann [10]. The equivalence of (2) and (3) has been proved in [14] and our proof of Theorem 1 is complete.

LEMMA 3.1. *For $1 \leqslant p \leqslant \infty$, $f \in D$, we have the estimate*

$$\|\beta_n(f) - f\|_p \leqslant A_p \|S(f)\|_p / n, \tag{3.1}$$

where A_p is a constant depending only on p.

Proof. Note that

$$f(t) - f(x) = \int_x^t (1/u - 1/t)(u^2 f'(u))'\, du + (1/x - 1/t)\, x^2 f'(x). \tag{3.2}$$

We have

$$\begin{aligned} |\beta_n(f, x) - f(x)| &\leqslant |x^2 f'(x)\, \beta_n(1/x - 1/t, x)| \\ &\quad + \beta_n\left(\left|\int_x^t (1/u - 1/t)(u^2 f'(u))'\, du\right|, x\right) \\ &\leqslant |x f'(x)|/(n-1) \\ &\quad + \beta_n\left(|1/x - 1/t| \left|\int_x^t |(u^2 f'(u))'|\, du\right|, x\right). \end{aligned} \tag{3.3}$$

Thus for $p=\infty$, we can estimate easily as follows,

$$\begin{aligned}\|\beta_n(f)-f\|_\infty &\leqslant \|\varphi f'\|_\infty/(n-1)\\ &\quad+(\|\varphi^2 f''\|_\infty+2\,\|\varphi f'\|_\infty)\,\beta_n\left(\frac{t^2-2tx+x^2}{tx}, x\right)\\ &\leqslant \|\varphi f'\|_\infty/(n-1)+2(\|\varphi^2 f''\|_\infty+2\,\|\varphi f'\|_\infty)/(n-1)\\ &\leqslant 14\,\|\varphi^2 f''\|_\infty/n.\end{aligned}$$

For $1<p<\infty$, we use the maximal function $M((\varphi^2 f')')(x)$ and obtain

$$\begin{aligned}\|\beta_n(f)-f\|_p &\leqslant \|\varphi f'\|_p/(n-1)\\ &\quad+\left\|M((\varphi^2 f')')(x)\,\beta_n\left(\frac{t^2-2tx+x^2}{tx}, x\right)\right\|_p\\ &\leqslant \|\varphi f'\|_p/(n-1)+2\,\|M((\varphi^2 f')')\|_p/(n-1)\\ &\leqslant \|\varphi^2 f''\|_p/(n-1)+2A'_p\,\|\varphi^2 f''\|_p/(n-1)\leqslant A_p\,\|S(f)\|_p/n.\end{aligned}$$

The proof for the case $p=1$ is somewhat different.
By Taylor's formula

$$f(t)-f(x)=(t-x)\,f'(x)+\int_x^t (t-u)\,f''(u)\,du:=I+J, \tag{3.4}$$

we can estimate by Fubibi's theorem as follows,

$$\|\beta_n(I, x)\|_{L^1[0,\infty)}=\|xf'(x)\|_1/(n-1)\leqslant 2\,\|S(f)\|_1/n; \tag{3.5}$$

$$\begin{aligned}\|\beta_n(J, x)\|_{L^1[0,\infty)} &= \int_0^\infty \left|\int_0^\infty b_n(x, t)\int_x^{1/t} (1/t-u)\,f''(u)\,du\,dt\right| dx\\ &\leqslant \int_0^\infty \left(\int_0^{1/x} b_n(x, t)\int_x^{1/t} (1/t-u)\,|f''(u)|\,du\,dt\right.\\ &\quad\left.+\int_{1/x}^\infty b_n(x, t)\int_{1/t}^x (u-1/t)\,|f''(u)|\,du\,dt\right) dx\\ &= \int_0^\infty dx\int_x^\infty |f''(u)|\,du\int_0^{1/u} b_n(x, t)(1/t-u)\,dt\\ &\quad+\int_0^\infty dx\int_0^x |f''(u)|\,du\int_{1/u}^\infty b_n(x, t)(u-1/t)\,dt\end{aligned}$$

$$=\int_0^\infty |f''(u)|\,du\left(\int_0^u\int_0^{1/u} b_n(x,t)(1/t-u)\,dt\,dx\right.$$

$$\left.+\int_u^\infty\int_{1/u}^\infty b_n(x,t)(u-1/t)\,dt\,dx\right)$$

$$=\int_0^\infty |f''(u)|\,du\left(\int_0^u\int_0^\infty b_n(x,t)(1/t-u)\,dt\,dx\right.$$

$$\left.+\int_0^\infty\int_{1/u}^\infty b_n(x,t)(u-1/t)\,dt\,dx\right)$$

$$=\int_0^\infty |f''(u)|\,du\left(\int_0^u (nx/(n-1)-u)\,dx\right.$$

$$\left.+\int_{1/u}^\infty (u-1/t)\,nt^{-2}/(n-1)\,dt\right)$$

$$=\int_0^\infty u^2\,|f''(u)|\,du/(n-1)\leqslant 2\,\|S(f)\|_1/n.$$

Thus our proof of Lemma 3.1 has been obtained.

LEMMA 3.2. *For* $1\leqslant p\leqslant\infty$, $f\in D$, $n\geqslant 2$, *we have*

$$\|S(\beta_n f)\|_p\leqslant 2\,\|S(f)\|_p. \tag{3.6}$$

Proof. By Lemma 2.4, we have for $\varphi^2 f''\in L^p[0,\infty)$

$$\|\varphi^2(\beta_n f)''\|_p\leqslant\|\beta_n(\varphi^2 f'')\|_p\leqslant n\,\|\varphi^2 f''\|_p/(n-1)\leqslant 2\,\|S(f)\|_p.$$

The following result is the so-called Bernstein-type inequality.

LEMMA 3.3. *For* $1\leqslant p\leqslant\infty$, $f\in L^p[0,\infty)$, $n\geqslant 2$, *we have*

$$\|S(\beta_n f)\|_p\leqslant 3n\,\|f\|_p. \tag{3.7}$$

Proof. By simple calculations we have

$$(\beta_n f)'(x)=\int_0^\infty\frac{f(1/u)}{B(n,n)}\,(nu^{n-1}x^{n-1}(1+xu)^{-2n}$$

$$-2nu^n x^n(1+xu)^{-2n-1})\,du, \tag{3.8}$$

$$x^2(\beta_n f)''(x) = \int_0^\infty \frac{f(1/u)}{B(n,n)} (n(n-1)\, u^{n-1} x^n (1+xu)^{-2n}$$
$$- 4n^2 u^n x^{n+1}/(1+xu)^{2n+1}$$
$$+ 2n(2n+1)\, u^{n+1} x^{n+2} (1+xu)^{-2n-2})\, du$$
$$= \int_0^\infty \frac{f(1/u)}{B(n,n)} n u^{n-1} x^n ((n+1)(xu-1)^2 - 2)(1+xu)^{-2n-2}\, du. \tag{3.9}$$

Thus for $1 < p < \infty$, we obtain

$$\int_0^\infty |x^2(\beta_n f)''|^p \, dx$$
$$\leqslant \int_0^\infty \left(\int_0^\infty \frac{|f(1/u)|}{B(n,n)} n u^{n-1} x^n \left(\frac{n+1}{(1+xu)^{2n}} \right.\right.$$
$$\left.\left. - \frac{4(n+1)\, ux - 2}{(1+xu)^{2n+2}} \right) du \right)^p dx$$
$$\leqslant \int_0^\infty \left(\int_0^\infty \left(n(n+1)\, b_n(x,u) - \frac{B(n+1, n+1)}{B(n,n)} n b_{n+1}(x,u) \right.\right.$$
$$\left.\left. \times (4n+4-2/(ux)) \right) du \right)^{p/q}$$
$$\times \int_0^\infty |f(1/u)|^p \left(n(n+1)\, b_n(x,u) - n^2 b_{n+1}(x,u) \right.$$
$$\left. \times \left(4n + 4 - \frac{2}{ux} \right) \Big/ (4n+2)\, du \right) dx$$
$$\leqslant (3n)^{p/q} \int_0^\infty |f(1/u)|^p (n(n+1)\, \beta_n(1/t, u)/u$$
$$- (4n+4)\, n^2 \beta_{n+1}(1/t, u)/(u(4n+2)) + 2n^2 \beta_{n+1}(1, u)/(u^2(4n+2)))\, du$$
$$\leqslant (3n)^p \int_0^\infty |f(1/u)|^p/u^2 \, du;$$

hence the Bernstein-type inequality

$$\|S(\beta_n f)\|_p \leqslant 3n \, \|f\|_p$$

holds.

The cases $p = 1$ and ∞ are trivial and the proof is complete.

ACKNOWLEDGMENTS

The author is very grateful to Professor Zhu-Rui Guo for his kind help and encouragement. Thanks are also due to Professor V. Totik for his comments.

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